

ZERO-DIVISOR GRAPHS OF LOWER DISMANTLABLE LATTICES-I

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Dedicated to Professor N. K. Thakare on his 77th birthday

ABSTRACT. In this paper, we study the zero-divisor graphs of a subclass of dismantlable lattices. These graphs are characterized in terms of the non-ancestor graphs of rooted trees.

Keywords: Dismantlable lattice, adjunct element, adjunct representation, zero-divisor graph, cover graph, incomparability graph.

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1. INTRODUCTION

Beck [3] introduced the concept of zero-divisor graph of a commutative ring R with unity as follows. Let G be a simple graph whose vertices are the elements of R and two vertices x and y are adjacent if $xy = 0$. The graph G is known as the *zero-divisor graph* of R . He was mainly interested in the coloring of this graph. This concept is well studied in algebraic structures such as rings, semigroups, lattices, semilattices as well as in ordered structures such as posets and qosets; see Anderson et al. [2], Alizadeh et al. [1], LaGrange [14, 15], Lu and Wu [16], Joshi and Khiste [9], Nimbhorkar et al. [17], Halaš and Jukl [5], Joshi [8], Joshi, Waphare and Pourali [11, 12] and Halaš and Länger [6].

A graph is called *realizable as zero-divisor graph* if it is isomorphic to the zero-divisor graph of an algebraic structure or an ordered structure. In [15], LaGrange characterized simple graphs which are realizable as the zero-divisor graphs of Boolean rings and in [16], Lu and Wu gave a class of graphs that are realizable as the zero-divisor graphs of posets. Recently, Joshi and Khiste [9] extended the result of LaGrange [15] by characterizing simple graphs which are realizable as zero-divisor graphs of Boolean posets.

In this paper, we provide a class of graphs, namely the non-ancestor graphs of rooted trees, that are realizable as zero-divisor graphs of lower dismantlable lattices. In fact we prove:

Theorem 1.1. *For a simple undirected graph G , the following statements are equivalent.*

- (a) $G \in \mathcal{G}_{\mathcal{T}}$, the class of non-ancestor graphs of rooted trees.
- (b) $G = G_{\{0\}}(L)$ for some lower dismantlable lattice L with the greatest element 1 as a join-reducible element.
- (c) G is the incomparability graph of $(L \setminus \{0, 1\}, \leq)$ for some lower dismantlable lattice L with the greatest element 1 as a join-reducible element.

Rival [18] introduced dismantlable lattices to study the combinatorial properties of doubly irreducible elements. By dismantlable lattice, we mean a lattice which can be completely “dismantled” by removing one element at each stage. Kelly and Rival [13] characterized dismantlable lattices by means of crowns, whereas Thakare, Pawar and Waphare [20] gave a structure theorem for dismantlable lattices using adjunct operation.

Now we begin with the necessary definitions and terminology.

Definition 1.2. A nonzero element p of a lattice L with 0 is an *atom* if $0 \prec p$ (by $a \prec b$, we mean there is no c such that $a < c < b$). Dually, a nonzero element d of a lattice L with 1 is a *dual atom* if $d \prec 1$.

Definition 1.3 (Definition 2.1, Thakare et al. [20]). If L_1 and L_2 are two disjoint finite lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\prec b$. Define the partial order \leq on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows.

- $x \leq y$ in L if
- either $x, y \in L_1$ and $x \leq y$ in L_1 ;
- or $x, y \in L_2$ and $x \leq y$ in L_2 ;
- or $x \in L_1, y \in L_2$ and $x \leq a$ in L_1 ;
- or $x \in L_2, y \in L_1$ and $b \leq y$ in L_1 .

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure of obtaining L in this way is called an *adjunct operation of L_2 to L_1* . The pair (a, b) is called an *adjunct pair* and L is an *adjunct* of L_2 to L_1 with respect to the adjunct pair (a, b) and we write $L = L_1]_a^b L_2$.

We place the Hasse diagrams of L_1, L_2 side by side in such a way that the greatest element 1_{L_2} of L_2 is at the lower position than b and the least element 0_{L_2} of L_2 is at the higher position than a . Then add the coverings $1_{L_2} \prec b$ and $a \prec 0_{L_2}$, as shown in Figure 1, to obtain the Hasse diagram of $L = L_1]_a^b L_2$.

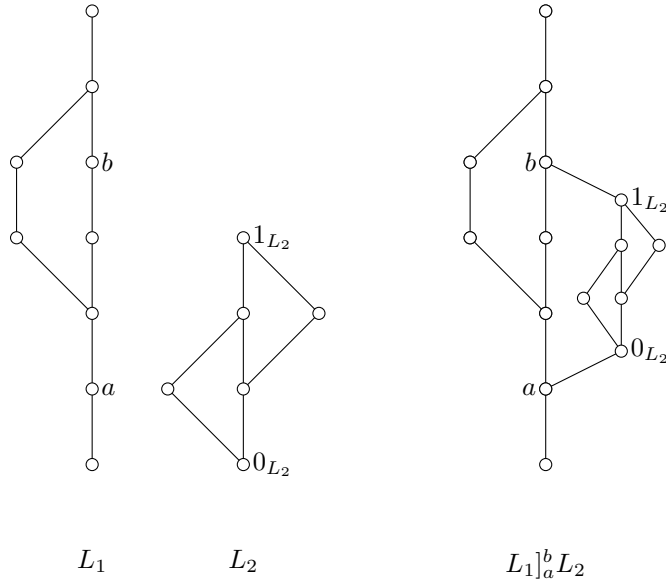


FIGURE 1. Adjunct of two lattices L_1 and L_2

Clearly, $|E(L)| = |E(L_1)| + |E(L_2)| + 2$, where $E(L)$ is nothing but edge set of L . This also implies that the adjunct operation preserves all the covering relations of the individual lattices L_1 and L_2 . Also note that if $x, y \in L_2$, then $a \prec 0_{L_2} \leq x \wedge y$. Hence $x \wedge y \neq 0$ in $L = L_1]_a^b L_2$.

2. PROPERTIES OF ZERO-DIVISOR GRAPHS OF DISMANTLABLE LATTICES

Following Beck [3], Nimbhorkar et al. [17] introduced the concept of zero-divisor graph of meet-semilattices with 0, which was further extended by Halaš and Jukl [5] to posets with 0. Recently, Joshi [8] introduced the zero-divisor graph with respect to an ideal I of a poset with 0. Note that his definition of zero-divisor graph coincides with the definition of Lu and Wu [16] when $I = \{0\}$.

A nonempty subset I of a lattice L is an *ideal* of L if $a, b \in I$ and $c \in L$ with $c \leq a$ implies $c \in I$ and $a \vee b \in I$. An ideal $I \neq L$ is a *prime ideal* if $a \wedge b \in I$ implies either $a \in I$ or $b \in I$. A prime ideal P of a lattice L is a *minimal prime ideal* if for any prime ideal Q we have $P \subseteq Q \subseteq L$ implies either $P = Q$ or $Q = L$.

Now, we recall the definition of zero-divisor graph given by Joshi [8] when the corresponding poset is a lattice and an ideal $I = \{0\}$.

Definition 2.1 (Definition 2.1, Joshi [8]). Let L be a lattice with the least element 0. We associate a simple undirected graph $G_{\{0\}}(L)$ as follows. The set of vertices of $G_{\{0\}}(L)$ is $V(G_{\{0\}}(L)) = \{x \in L \setminus \{0\} : x \wedge y = 0 \text{ for some } y \in L \setminus \{0\}\}$ and distinct vertices x, y are adjacent if and only if $x \wedge y = 0$. The graph $G_{\{0\}}(L)$ is called the *zero-divisor graph* of L .

The following Figure 2 illustrates the zero-divisor graph $G_{\{0\}}(L)$ of the given lattice L .

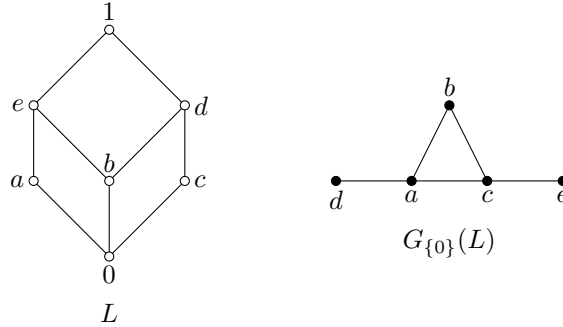


FIGURE 2. A lattice L with its zero-divisor graph $G_{\{0\}}(L)$

Note that $\{0\}$ is a prime ideal in a lattice L with 0 if and only if $V(G_{\{0\}}(L)) = \emptyset$.

Now, we reveal the structure of zero-divisor graph of $L = L_1]_a^b L_2$ in terms of zero-divisor graphs of L_1 and L_2 . For that purpose, we need the following definitions.

Definition 2.2. Given two graphs G_1 and G_2 , the *union* $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2 , denoted by $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{\{x_1, x_2\} \mid x_1 \in V(G_1), x_2 \in V(G_2)\}$. The *null graph* on a set S is the graph whose vertex set is S and edge set is the empty set, we denote it by $N(S)$.

Throughout this paper all the lattices are finite.

The following result describes the zero-divisor graph of adjunct of two lattices.

Theorem 2.3. Let L_1 and L_2 be two lattices. Put $L = L_1]_a^b L_2$. Then the following statements are true.

- (a) If $a \neq 0$ and $a \notin V(G_{\{0\}}(L_1))$, then $G_{\{0\}}(L) = G_{\{0\}}(L_1)$.
- (b) If $a \in V(G_{\{0\}}(L_1))$, then $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (G_a + N(L_2))$, where $G_a = \{x \in L_1 \mid a, x \text{ are adjacent in } G_{\{0\}}(L_1)\}$.

- (c) If $a = 0$, then $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup \left(N(L_1^* \setminus [b]) + N(L_2) \right)$, where $[b] = \{x \in L_1 : x \geq b\}$ is the principal dual ideal generated by b in L_1 and $L_1^* = L_1 \setminus \{0\}$.

Proof. (a) Let $a \neq 0$ and $a \notin V(G_{\{0\}}(L_1))$. If $x \in V(G_{\{0\}}(L))$ is adjacent to some $y \in L_2$, then $x \wedge y = 0$, clearly $x \notin L_2$. As $a \leq y$ in L , we get $a \wedge x = 0$. Hence $a \in V(G_{\{0\}}(L_1))$, a contradiction to the fact that $a \notin V(G_{\{0\}}(L_1))$. Hence no element of L_2 is adjacent to any vertex of $G_{\{0\}}(L)$. Also, $G_{\{0\}}(L_1)$ is a subgraph of $G_{\{0\}}(L)$, therefore $G_{\{0\}}(L) = G_{\{0\}}(L_1)$.

(b) Now, let $a \in V(G_{\{0\}}(L_1))$. If $x \in V(G_{\{0\}}(L))$, then there exists a nonzero element $y \in L$ such that $x \wedge y = 0$. This implies that at most one of x and y may be in L_2 , otherwise $a \leq x \wedge y = 0$, a contradiction. If $x, y \in L_1$, then $x \in V(G_{\{0\}}(L_1))$. Without loss of generality, let $x \in L_1$ and $y \in L_2$, which gives $x \wedge a = 0$, since $a \leq y$. Therefore $x \in G_a$.

Thus $V(G_{\{0\}}(L)) \subseteq V(G_{\{0\}}(L_1) \cup (G_a + N(L_2)))$.

Since $V(G_{\{0\}}(L_1) \cup (G_a + N(L_2))) \subseteq V(G_{\{0\}}(L))$, the equality holds.

Let x and y be adjacent in $G_{\{0\}}(L)$. Hence at most one of x and y may be in L_2 . If $x, y \in L_1$, then x and y are adjacent in $G_{\{0\}}(L_1)$. Now, without loss of generality, assume that $x \in L_1$ and $y \in L_2$. Therefore $x \wedge a = 0$. Hence $x \in G_a$, i.e., x and y are adjacent in $G_a + N(L_2)$.

Now, let x and y be adjacent in $G_{\{0\}}(L_1) \cup (G_a + N(L_2))$. If $x, y \in G_{\{0\}}(L_1)$, we are through. Let $x \in G_a$ and $y \in L_2$. Then $x \wedge a = 0$. We claim that $x \wedge y = 0$. Suppose $x \wedge y \neq 0$. Then we have two possibilities either $x \wedge y \in L_1$ or $x \wedge y \in L_2$. If $x \wedge y \in L_2$, then by the definition of adjunct, we have $a \leq x \wedge y$ which yields $a \leq x \wedge y \wedge a = 0$, a contradiction. Thus $x \wedge y \notin L_2$. Hence $x \wedge y \in L_1$. Since $x \wedge y \leq y$ for $y \in L_2$, again by the definition of adjunct, we have $x \wedge y \leq a$. This gives $x \wedge y \wedge x \leq a \wedge x = 0$, a contradiction to the fact that $x \wedge y \neq 0$. Thus we conclude that $x \wedge a = 0$ if and only if $x \wedge y = 0$ for any $y \in L_2$. Therefore $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (G_a + N(L_2))$.

(c) Assume that $a = 0$. Let $x \in V(G_{\{0\}}(L))$. Then there exists a nonzero element $y \in L$ such that $x \wedge y = 0$. Hence at most one of x and y may be in L_2 . If $x, y \in L_1$, then $x \in G_{\{0\}}(L_1)$. Without loss of generality, let $x \in L_1$ and $y \in L_2$. If $x \geq b$, then by the definition of adjunct, we have $x \geq y$, a contradiction to the fact that $x \wedge y = 0$. Hence $x \not\geq b$, i.e., $x \in N(L_1^* \setminus [b])$. Therefore $x \in V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2)))$.

Now, assume that $x \in V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2)))$.

If $x \in G_{\{0\}}(L_1)$, then we are through. Let $x \in V(N(L_1^* \setminus [b]) + N(L_2))$. If $x \in N(L_1^* \setminus [b])$, then $x \not\geq b$. For any $y \in L_2$, we have $x \parallel y$ and by the definition of adjunct and $a = 0$, we have $x \wedge y = 0$. Therefore $x \in V(G_{\{0\}}(L))$, as y is a nonzero element of L .

If $x \in N(L_2)$, then for any atom p of L_1 , $p \wedge x = 0$. Therefore $x \in G_{\{0\}}(L)$. Hence we have $V(G_{\{0\}}(L)) = V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2)))$.

Let x and y be adjacent in $G_{\{0\}}(L)$. Then at most one of x and y may be in L_2 . If $x, y \in L_1$, then they are adjacent in $G_{\{0\}}(L_1)$. Therefore they are adjacent in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2))$. Without loss of generality, let $x \in L_1$ and $y \in L_2$. As above, $x \not\geq b$. Hence $x \in (N(L_1^* \setminus [b])$ and $y \in L_2$. Therefore they are adjacent in $N(L_1^* \setminus [b]) + N(L_2)$ and hence in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2))$.

Conversely, suppose that x and y are adjacent in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2))$. If both $x, y \in G_{\{0\}}(L_1)$, we are done. Also, at most one of x and y may be in L_2 . Without loss of generality, let $x \in L_1$ and $y \in L_2$, then $x \not\geq b$, i.e., $x \in N(L_1^* \setminus [b])$. Therefore $x \wedge y = 0$ in L . Hence x and y are adjacent in $G_{\{0\}}(L)$. Therefore we get $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (N(L_1^* \setminus [b]) + N(L_2))$. \square

The following Figures 3, 4 and 5 illustrate Theorem 2.3.

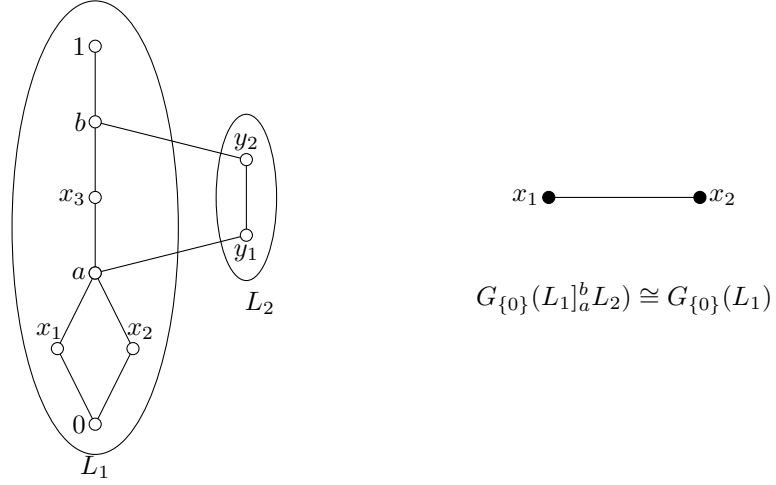


FIGURE 3. Theorem 2.3, Case (a)

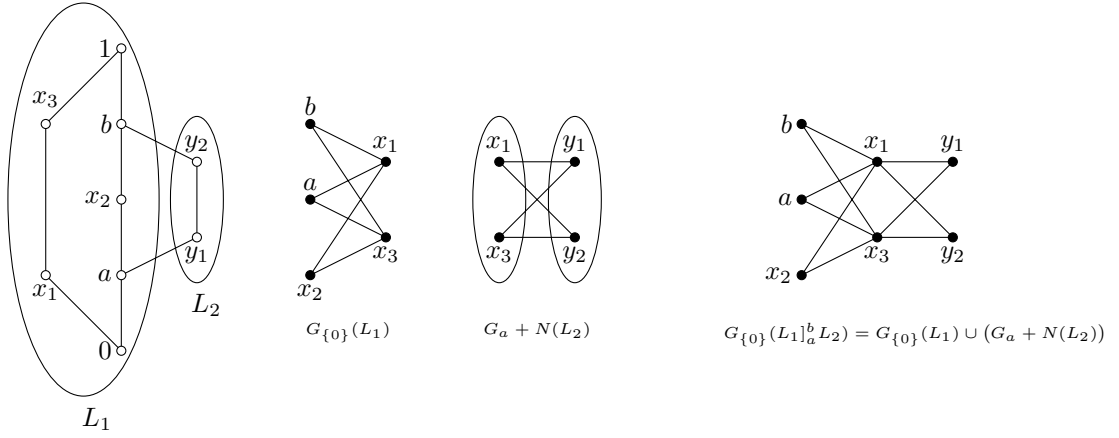


FIGURE 4. Theorem 2.3, Case (b)

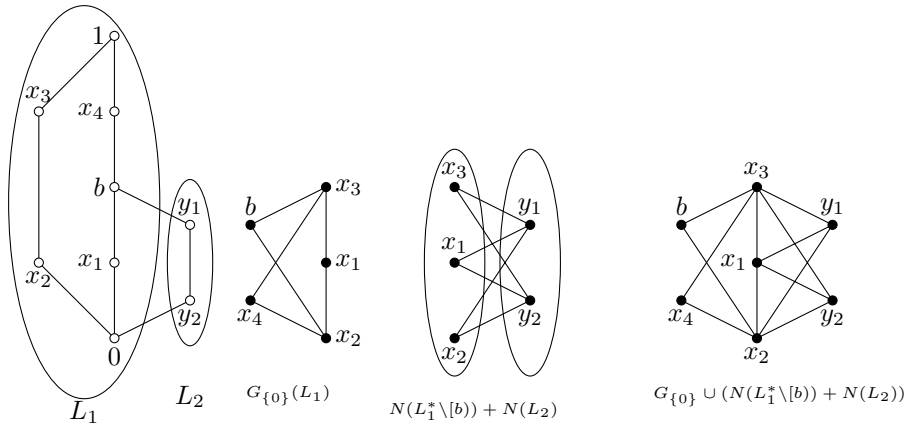


FIGURE 5. Theorem 2.3, Case (c)

Let G be a graph and x, y be distinct vertices in G . A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. If G has a x, y -path, then the *distance* from x to y , written $d(x, y)$, is the least length of a x, y -path. If G has no such path, then $d(x, y) = \infty$. The *diameter* ($\text{diam}(G)$) is $\max_{x, y \in V(G)} d(x, y)$; see West [21].

Corollary 2.4. Let L be an adjunct of two chains C_1 and C_2 with an adjunct pair $(a, 1)$, i.e., $L = C_1]_a^1 C_2$. If $G_{\{0\}}(L) \neq \emptyset$, then $a = 0$ and $G_{\{0\}}(L)$ is a complete bipartite graph. Hence $\text{diam}(G_{\{0\}}(L)) \leq 2$. Moreover, $G_{\{0\}}(L) = K_{m,n}$, if $|C_1| = n + 2$ and $|C_2| = m$ where $m, n \in \mathbb{N}$.

Proof. Let $L = C_1]_a^1 C_2$ and $G_{\{0\}}(L) \neq \emptyset$. Clearly $V(G_{\{0\}}(C_1)) = \emptyset$. If $a \neq 0$, then by Theorem 2.3(a), we have $G_{\{0\}}(L) = G_{\{0\}}(C_1) = \emptyset$, a contradiction. Therefore $a = 0$.

Now, every element of $C_1 \setminus \{0, 1\}$ is adjacent to each element of C_2 . Hence $G_{\{0\}}(L)$ is a complete bipartite graph, in fact $G_{\{0\}}(L) = K_{m,n}$ whenever $|C_1| = n + 2$ and $|C_2| = m$ where $m, n \in \mathbb{N}$. \square

Notation. Let $\mathcal{M}_n = \{0, 1, a_1, a_2, \dots, a_n\}$ be a lattice such that $0 < a_i < 1$, for every i , $i = 1, 2, \dots, n$ with $a_i \wedge a_j = 0$ and $a_i \vee a_j = 1$ for every $i \neq j$.

Remark 2.5. If L is an adjunct of more than two chains, then $G_{\{0\}}(L)$ need not be bipartite. Consider the lattice $L = \mathcal{M}_3$ depicted in Figure 6. Consider $L = C_1]_0^1 C_2]_0^1 C_3$ where $C_1 = \{0, x, 1\}$, $C_2 = \{y\}$ and $C_3 = \{z\}$. Then $G_{\{0\}}(L) \cong K_3$, a non bipartite graph.

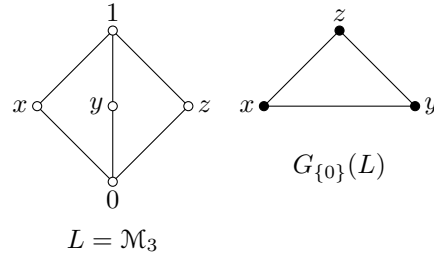


FIGURE 6. A dismantlable lattice with non-bipartite zero-divisor graph

The concept of dismantlable lattice was introduced by Rival [18].

Definition 2.6 (Rival [18]). A finite lattice L having n elements is called *dismantlable*, if there exists a chain $L_1 \subset L_2 \subset \dots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i .

The following structure theorem is due to Thakare, Pawar and Waphare [20].

Structure Theorem (Theorem 2.2, Thakare et al. [20]). A finite lattice is dismantlable if and only if it is an adjunct of chains.

From the above structure theorem and Corollary 2.4, it is clear that if the vertex set of a zero-divisor graph of adjunct of two chains is nonempty then it is a lower dismantlable lattice in the following sense.

Definition 2.7. We call a dismantlable lattice L to be a *lower dismantlable* if it is a chain or every adjunct pair in L is of the form $(0, b)$ for some $b \in L$.

It should be noted that any lattice of the form $L = C_0]_0^{x_1} C_1]_0^{x_2} \dots]_0^{x_n} C_n$ is always a lower dismantlable lattice, where C_i 's are chains. Consider the lattices depicted in Figure 7. Observe that the lattice L is lower dismantlable whereas the lattice L' is not lower dismantlable.

Definition 2.8. An element x in a lattice L is *join-reducible* (*meet-reducible*) in L if there exist $y, z \in L$ both distinct from x , such that $y \vee z = x$ ($y \wedge z = x$); x is *join-irreducible* (*meet-irreducible*) if it is not join-reducible (*meet-reducible*); x is *doubly irreducible* if it is both join-irreducible and meet-irreducible. Therefore, an element x is doubly reducible in a lattice L if and only if x has

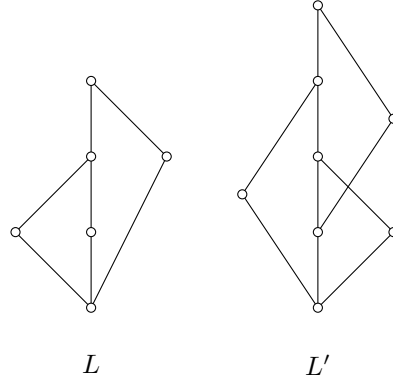
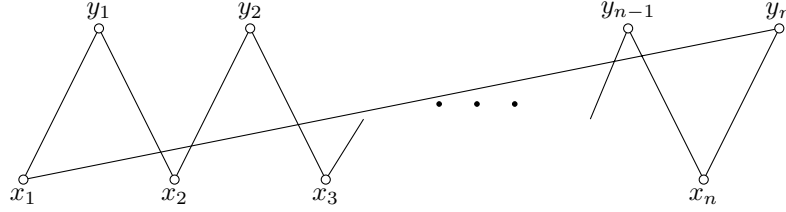


FIGURE 7. Examples of lower dismantlable and non- lower dismantlable lattice

at most one lower cover or x has at most one upper cover. The set of all meet-irreducible (join-irreducible) elements in L is denoted by $M(L)$ ($J(L)$). The set of all doubly irreducible elements in L is denoted by $Irr(L)$ and its in L is denoted by $Red(L)$. Thus, if $x \in Red(L)$ then x is either join-reducible or meet-reducible.

For an integer $n \geq 3$, a *crown* is a partially ordered set $\{x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$ in which $x_i \leq y_i$, $x_{i+1} \leq y_i$, for $i = 1, 2, \dots, n-1$ and $x_1 \leq y_n$ are the only comparability relations (see Figure 8).

FIGURE 8. Crown of order $2n$

Note that if L is lower dismantlable lattice with the greatest element 1 as a join-reducible element, then it is easy to observe that every nonzero nonunit element of L is a vertex of $G_{\{0\}}(L)$.

The following lemma gives the properties of lower dismantlable lattices which will be used in the sequel frequently.

Lemma 2.9. *Let $L = C_0]_0^{x_1} C_1]_0^{x_2} \dots]_0^{x_n} C_n$ be a lower dismantlable lattice, where C_i 's are chains. Then for nonzero elements $a, b \in L$, we have.*

- i) $a \wedge b = 0$ if and only if $a \parallel b$ (where $a \parallel b$ means a and b are incomparable).
- ii) Let $a \in C_i$ with $b \notin C_i$, then $a \leq b$ if and only if $x_i \leq b$.
- iii) If $(0, 1)$ is an adjunct pair (i.e., $x_i = 1$ for some $i \in \{1, 2, \dots, n\}$), then $|V(G_{\{0\}}(L))| = |L| - 2$.

Proof. i) Suppose $a \parallel b$ and $a \wedge b \neq 0$. It is clear that there is an adjunct pair (a_1, b_1) in the adjunct representation of L such that $a_1 = a \wedge b \neq 0$, a contradiction to the definition of lower dismantlability of L . The converse is obvious.

ii) Let $a \in C_i$ and $b \notin C_i$ with $x_i \leq b$. As C_i is joined at x_i , we must have $a \leq x_i$. Hence $a \leq b$. Conversely, suppose $a \leq b$. Now, $a \in C_i$ and $b \notin C_i$. If $x_i \not\leq b$, we have either $b \leq x_i$ or $x_i \parallel b$. The second case is impossible, by (i) above, as it gives $a \wedge b = 0$, since $a \leq x_i$. Also, it is given that $b \notin C_i$, hence $a \parallel b$ which yields $a \wedge b = 0$, a contradiction to the fact that $a \leq b$ and $a \neq 0$. Therefore

$x_i \leq b$.

iii) As L is a lower dismantlable lattice having $(0, 1)$ as an adjunct pair, L contains at least two chains in its adjunct representation. Also $a \wedge b = 0$ if and only if $a \parallel b$. Hence any $a \in L \setminus \{0, 1\}$ is in $V(G_{\{0\}}(L))$. and consequently $|V(G_{\{0\}}(L))| = |L| - 2$. \square

Now, we recall some definitions from graph theory.

Definition 2.10. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The *girth* of a graph with cycle, written $gr(G)$, is the length of its shortest cycle. A graph with no cycle has infinite girth. A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A tree is called a *rooted tree* if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root. A vertex w of a rooted tree is called an *ancestor* of v if w is on the unique path from v to the root of the tree; see West [21].

Let T be a rooted tree with the root R has at least two branches. Let $G(T)$ be the *non-ancestor graph* of T , i.e., $V(G(T)) = T \setminus \{R\}$ and two vertices are adjacent if and only if no one is an ancestor of the other. Denote the class of non-ancestor graphs of rooted trees by \mathcal{G}_T . The *cover graph* of a lattice L , denoted by $CG(L)$, is the graph whose vertices are the elements of L and whose edges are the pairs (x, y) with $x, y \in L$ satisfying $x \prec y$ or $y \prec x$. The *comparability graph* of a lattice L , denoted by $C(L)$, is the graph whose vertices are the elements of L and two vertices x and y are adjacent if and only if x and y are comparable. The complement of the comparability graph $C(L)$, i.e., $C(L)^c$, is called the *incomparability graph* of L .

The following result is due to Kelly and Rival [13].

Theorem 2.11 (Theorem 3.1, Kelly and Rival [13]). *A finite lattice is dismantlable if and only if it contains no crown.*

In the following theorem, we characterize the zero-divisor graph $G_{\{0\}}(L)$ of a lower dismantlable lattice L in terms of the cover graph $CG(L)$ and the incomparability graph $C(L)^c$.

Theorem 2.12. *The following statements are equivalent for a finite lattice L with 1 as a join-reducible element.*

- (a) L is a lower dismantlable lattice.
- (b) Every nonzero element of L is a meet-irreducible element.
- (c) The cover graph $CG(L \setminus \{0\})$ of $L \setminus \{0\}$ is a tree.
- (d) The zero-divisor graph of L coincides with the incomparability graph of $L \setminus \{0, 1\}$.

Proof. (a) \Rightarrow (b) Let L be a lower dismantlable lattice. Then by Structure Theorem, $L = C_1]_0^{a_1} C_2]_0^{a_2} \cdots]_0^{a_{n-1}} C_n$, where each C_i is a chain. Let $a (\neq 0) \in L$ be an element which is not meet-irreducible. Then $a = b \wedge c$ for some $b, c \neq a$. But then $b \parallel c$. By Lemma 2.9, $a = b \wedge c = 0$, a contradiction to $a \neq 0$. Hence every nonzero element of L is meet irreducible element.

(b) \Rightarrow (c) Let $CG(L \setminus \{0\})$ be the cover graph of $L \setminus \{0\}$ and let $C : a_1 - a_2 - \cdots - a_n - a_1$ be a cycle in $CG(L \setminus \{0\})$. For distinct a_1, a_2, a_3 we have the following three cases.

Case (1): Let $a_1 \prec a_2 \prec a_3$ be a chain. Then for a_4 , we have either $a_1 \prec a_2 \prec a_3 \prec a_4$ is a chain or $a_1 \prec a_2 \prec a_3$ and $a_4 \prec a_3$. Then, $a_{i+1} \leq a_i$, for all $i \geq 4$; otherwise we get a_4 as a meet reducible element. Hence $a_n \leq a_{n-1}$ and $a_n = a_{n-1} \wedge a_1 \neq 0$, as $C : a_1 - a_2 - \cdots - a_n - a_1$ is a cycle. This contradicts the fact that every nonzero element is meet-irreducible. On the other hand, if $a_1 \prec a_2 \prec a_3 \prec a_4$ is chain, then using the above arguments, we have $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5$ is

a chain, Continuing in this way we get $a_1 \prec a_2 \prec \cdots \prec a_n$ is a chain and $a_1 \leq a_n$. Hence a_1 and a_n can not be adjacent, a contradiction to the fact that C is a cycle in $CG(L \setminus \{0\})$.

Case (2): $a_2 = a_1 \wedge a_3$, which is impossible, as $a_1 \wedge a_3 = a_2 \neq 0$, a contradiction to lower dismantlability of L .

Case (3): $a_2 = a_3 \vee a_1$. Note that $a_3 \not\leq a_4$ otherwise a_3 is the meet of a_2 and a_4 , a contradiction to the fact that every nonzero element is meet-irreducible. Hence $a_4 \leq a_3$. In fact $a_{m+1} \leq a_m$, for $m \geq 3$. Using the above arguments, we again obtain a contradiction. Hence $CG(L \setminus \{0\})$ is a connected acyclic graph. Therefore it is a tree.

(c) \Rightarrow (a) If L contains a crown, then $CG(L \setminus \{0\})$ contains a cycle, a contradiction. Hence L does not contain a crown. By applying Theorem 2.11, L is a dismantlable lattice. Now, let a and b be incomparable elements of L . Suppose that $a \wedge b \neq 0$. Let $a \wedge b = a_1 \prec a_2 \prec \cdots \prec a_i = a \prec \cdots \prec a_n = a \vee b$, be a covering and also $a \wedge b = b_1 \prec b_2 \prec \cdots \prec b_j = b \prec \cdots \prec b_m = a \vee b$, be another covering, distinct from the first covering (such coverings exist, since $a \parallel b$). Then $a_1 - a_2 - \cdots - a_n = b_m - b_{m-1} - \cdots - b_1 = a_1$ is a cycle in $CG(L \setminus \{0\})$, a contradiction to the fact that $CG(L \setminus \{0\})$ is a tree. Thus L is a lower dismantlable lattice.

(d) \Rightarrow (b) Suppose $G_{\{0\}}(L) = C(L \setminus \{0, 1\})^c$, for some lattice L . We want to show that L does not contain any nonzero meet-reducible element. Suppose on the contrary, L has a nonzero meet-reducible element say b . Then there exist $a, c \in L$ with $a, c \neq b$, such that $b = a \wedge c$. Thus a and c are incomparable. So there is an edge $a - c$ in $C(L \setminus \{0, 1\})^c$. But $a \wedge c \neq 0$, hence a and c are not adjacent in $G_{\{0\}}(L)$, a contradiction. Consequently every nonzero element of L is a meet-irreducible element.

(b) \Rightarrow (d) Suppose every nonzero element of L is meet irreducible. By the equivalence of (a) and (b) above, L is a lower dismantlable lattice. Hence $a \wedge b = 0$ if and only if $a \parallel b$. Therefore $G_{\{0\}}(L) = C(L \setminus \{0, 1\})^c$. \square

Note that a result similar to the equivalence of statements (b) and (d) of Theorem 2.12 can be found in Survas [19].

Grillet and Varlet [4] introduced the concept of 0-distributive lattices as a generalization of distributive lattices. A lattice L with 0 is called *0-distributive* if, for every triplet (a, b, c) of elements of L , $a \wedge b = a \wedge c = 0$ implies $a \wedge (b \vee c) = 0$. More details about 0-distributive posets can be found in Joshi and Waphare [10]. Forbidden configurations for 0-distributive lattices are obtained by Joshi [7].

Lemma 2.13. *If L is an adjunct of two chains with $(0, 1)$ as an adjunct pair, then L is 0-distributive.*

Proof. Let $a \wedge b = a \wedge c = 0$, for $a, b, c \in L$. By Lemma 2.9, b and c are either comparable or $b \wedge c = 0$. If $b \wedge c = 0$, this together with $a \wedge b = a \wedge c = 0$ gives L as adjunct of at least three chains, a contradiction. Hence b and c are comparable. By $a \wedge b = a \wedge c = 0$, we have $a \wedge (b \vee c) = 0$. \square

The following result is due to Joshi [8].

Theorem 2.14 (Theorem 2.14, Joshi [8]). *Let L be a 0-distributive lattice. Then $G_{\{0\}}(L)$ is complete bipartite if and only if there exist two minimal prime ideals P_1 and P_2 of L such that $P_1 \cap P_2 = \{0\}$.*

Theorem 2.15. *Let L be a lower dismantlable lattice having $(0, 1)$ as an adjunct pair. Then the following statements are equivalent.*

- (a) $G_{\{0\}}(L)$ is a complete bipartite graph.
- (b) L is an adjunct of two chains only.
- (c) L has exactly two atoms and exactly two dual atoms.

- (d) *There exist two minimal prime ideals P_1 and P_2 of L such that $P_1 \cap P_2 = \{0\}$.*
- (e) *L is a 0-distributive lattice.*

Proof. (a) \Rightarrow (b) Suppose $G_{\{0\}}(L) = K_{m,n}$. Note that the independent set of $K_{m,n}$ forms a chain in L . If L is an adjunct of more than two chains, then L contains at least three atoms, which forms a triangle in $G_{\{0\}}(L)$, a contradiction. Hence L is adjunct of two chains only. Moreover, as $(0, 1)$ is an adjunct pair, $L = C_1]_0^1 C_2$, where C_1 and C_2 are chains.

(b) \Rightarrow (c) Obvious.

(c) \Rightarrow (d) Let d_1 and d_2 be only dual atoms of L . Then $P_1 = (d_1] = \{x \in L : x \leq d_1\}$ and $P_2 = (d_2]$ are ideals of L . In fact, $P_1 \cap P_2 = \{0\}$, otherwise we get a nonzero meet-reducible element in L , a contradiction to Theorem 2.12. We claim that P_1 and P_2 are prime ideals. Let $a \wedge b \in P_1$, i.e., $a \wedge b \leq d_1$. If $a \wedge b \neq 0$, then a and b are comparable. Hence $a \in P_1$ or $b \in P_1$. Suppose $a \wedge b = 0$ and $a, b \notin P_1 = (d_1]$. Then by Lemma 2.9, $a \parallel b$. Since d_2 is the only dual atom other than d_1 , we have $a, b \leq d_2$, which gives $(0, d_2)$ is an adjunct pair in L , hence there exist two atoms below d_2 , i.e., L contains three atoms (one below d_1 and two below d_2), a contradiction. Therefore P_1 is a prime ideal. Similarly P_2 is a prime ideal.

(d) \Rightarrow (a) Follows from Theorem 2.14 [8].

(a) \Rightarrow (e) Let $G_{\{0\}}(L)$ be a complete bipartite graph. By the equivalence of statements (a) and (b), L is an adjunct of exactly two chains. By Lemma 2.13, L is 0-distributive.

(e) \Rightarrow (a) Suppose that L is 0-distributive. Assume on the contrary, assume that L is an adjunct of more than two chains, i.e., L contains at least three atoms, say a, b, c . Clearly $a \wedge b = b \wedge c = a \wedge c = 0$. We consider the following two cases.

Case (1) : If $a \vee b = a \vee c = b \vee c$, then L contains a sublattice isomorphic to \mathcal{M}_3 (as shown in Figure 6), a contradiction to 0-distributivity of L .

Case (2) : Let $a \vee b \parallel b \vee c$. Clearly $b \leq (a \vee b) \wedge (b \vee c)$ is a nonzero meet-reducible element in L , a contradiction to Theorem 2.12. Hence $a \vee b$ and $b \vee c$ are comparable. Without loss of generality, suppose $a \vee b \leq b \vee c$. Hence $a \leq a \vee b \leq b \vee c$, which gives $a \wedge (b \vee c) = a$, but $a \wedge b = 0$ and $a \wedge c = 0$, again a contradiction to 0-distributivity of L .

Thus in any case L does not contain three atoms. Since $(0, 1)$ as an adjunct pair, it is clear that L is an adjunct of exactly two chains. Therefore $G_{\{0\}}(L)$ is a complete bipartite graph. \square

The following result is due to Joshi [8].

Theorem 2.16 (Theorem 2.4, Joshi [8]). *Let L be a lattice. Then $G_{\{0\}}(L)$ is connected with $\text{diam}(G_{\{0\}}(L)) \leq 3$.*

The following result can be found in Alizadeh et al. [1].

Theorem 2.17 (Theorem 4.2, Alizadeh [1]). *Let L be a lattice, then $\text{gr}(G_{\{0\}}(L)) \in \{3, 4, \infty\}$.*

In the following theorem, we characterize the diameter and girth of $G_{\{0\}}(L)$ for a lower dismantlable lattice L .

Theorem 2.18. *Let L be a lower dismantlable lattice which is an adjunct of n chains, where $n \geq 2$. Then $V((G_{\{0\}}(L))) \neq \emptyset$ and $\text{diam}(G_{\{0\}}(L)) \leq 2$. Moreover if $(0, 1)$ is the only adjunct pair in L , then $\text{diam}(G_{\{0\}}(L)) = 1$ if and only if $L \cong \mathcal{M}_n$. Further*

- (a) *$\text{gr}(G_{\{0\}}(L)) = 3$ if and only if L is an adjunct of at least three chains.*
- (b) *$\text{gr}(G_{\{0\}}(L)) = 4$ if and only if $L = C_1]_0^a C_2$ with $|C_2| \geq 2$.*
- (c) *$\text{gr}(G_{\{0\}}(L)) = \infty$ if and only if $L = C_1]_0^a C_2$ with $|C_2| = 1$.*

Proof. Let $a, b \in V((G_{\{0\}}(L)))$. If $a||b$, then by Lemma 2.9, $a \wedge b = 0$ and hence a, b are adjacent. Further let a and b are comparable, say $a \leq b$. Since $b \in V((G_{\{0\}}(L)))$, there is an element $c (\neq 0)$ such that $b \wedge c = 0$. Hence $a \wedge c = 0$. Thus we get a path $a - c - b$ of length 2. This shows that $d(a, b) \leq 2$ and in any case $\text{diam}(G_{\{0\}}(L)) \leq 2$.

Let $(0, 1)$ be the only adjunct pair in L . If $L \cong \mathcal{M}_n$, then $G_{\{0\}}(L) \cong K_n$. Hence $\text{diam}(G_{\{0\}}(L)) = 1$. Conversely, suppose $\text{diam}(G_{\{0\}}(L)) = 1$. If $L = C_1]_0^1 C_2$, then $G_{\{0\}}(L) \neq \emptyset$ if and only if $|C_1| \geq 3$ and $|C_2| \geq 1$. If $|C_1| \geq 3$ and $|C_2| = 2$ then for any $a, b \in C_2$ we have $d(a, b) = 2$. Now, If $|C_1| \geq 4$ and $|C_2| = 1$ then for any $a, b \in C_1 \setminus \{0, 1\}$, again $d(a, b) = 2$, a contradiction to the fact that $\text{diam}(G_{\{0\}}(L)) = 1$. Therefore $|C_1| = 3$ and $|C_2| = 1$ and the result follows by induction.

By Theorem 2.17, $gr(G_{\{0\}}(L)) \in \{3, 4, \infty\}$.

(a) If L is an adjunct of at least three chains, then it contains at least three atoms. Hence $G_{\{0\}}(L)$ contains a triangle, and $gr(G_{\{0\}}(L)) = 3$. Conversely, let $gr(G_{\{0\}}(L)) = 3$. If $L = C_1]_0^a C_2$, then by Corollary 2.4, $G_{\{0\}}(L)$ is complete bipartite. Therefore it does not contain an odd cycle, a contradiction. Hence L is adjunct of at least three chains.

(b) and (c) If $L = C_1]_0^a C_2$, then by Corollary 2.4, $G_{\{0\}}(L)$ is a complete bipartite graph. Then $|C_2| = 1$ or $|V(G_{\{0\}}(L)) \cap C_1| = 1$ if and only if $gr(G_{\{0\}}(L)) = \infty$. If $|C_2| \geq 2$ or $|V(G_{\{0\}}(L)) \cap C_1| \geq 2$, then $gr(G_{\{0\}}(L)) = 4$. \square

Remark 2.19. Note that if we drop the condition of lower dismantlability of L , then $\text{diam}(G_{\{0\}}(L))$ may exceed 2. Consider a lattice $L = C_1]_0^e C_2]_b^1 C_3]_0^d C_4$, where $C_1 = \{0, a, e, 1\}$, $C_2 = \{b\}$, $C_3 = \{d\}$ and $C_4 = \{c\}$. Then $\text{diam}(G_{\{0\}}(L)) = 3$ as shown in Figure 2.

Now, we give a realization of zero-divisor graphs of lower dismantlable lattices, *i.e.*, we describe graphs that are the zero-divisor graphs of lower dismantlable lattices.

Proof of Theorem 1.1. (a) \Rightarrow (b) Let $G \in \mathcal{G}_{\mathcal{T}}$. Hence $G = V(T \setminus \{R\})$ for some rooted tree T with the root R . Let $L = V(G) \cup \{R\} \cup \{0\}$. Define a relation \leq on L by, $a \leq R$, $0 \leq a$ and $a \leq a$, for every $a \in L$. If $a \neq b$, then $a < b$ if and only if b is an ancestor of a . Clearly, (L, \leq) is a poset. If $a||b$, then no one is ancestor of the other, hence 0 is the only element below a and b , *i.e.*, $a \wedge b = 0$. Let $A = \{c \in L \mid c \text{ is common ancestor of } a \text{ and } b\}$. Then $A \neq \emptyset$, as $R \in A$. We claim that the set A forms a chain. Let $x, y \in A$ with $x||y$. Hence x and y are ancestors of a and b both. But then $a - x - b - y - a$ is a cycle in the undirected graph of a rooted tree, a contradiction. Thus A is a chain. Then the smallest element of A (it exists due to finiteness of L) is nothing but $a \vee b$. Hence L is a lattice with the greatest element R , now denoted by 1. Since meet of any two incomparable elements is zero, L does not contain a crown. Hence by Theorem 2.11, L is a dismantlable lattice, say $L = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} \dots]_{a_n}^{b_n} C_n$. Since meet of any two incomparable elements is zero, we get $a_i = 0$, $\forall i$. Therefore L is a lower dismantlable lattice having 1 as join-reducible element (since the root of tree has at least two branches, hence 1 is join-reducible). Also, $a \wedge b = 0$ if and only if no one is an ancestor of the other. Therefore $G = G_{\{0\}}(L)$.

(b) \Rightarrow (c) Follows by Theorem 2.12.

(c) \Rightarrow (a) Let L be a lower dismantlable lattice. Let G be an incomparability graph of $L \setminus \{0, 1\}$, *i.e.*, $G = C(L \setminus \{0, 1\})^c$. Then Theorem 2.12, the cover graph of $L \setminus \{0\}$ is a rooted tree, say T , with a root 1. Let H be the non-ancestor graph of T . Then clearly, $V(H) = V(G)$ and a and b are adjacent in G if and only if $a||b$ if and only if no one is ancestor of the other if and only if a and b are adjacent in H . Hence $G = H$. \square

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